International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN 2319-3972
Vol. 2, Issue 4, Sep 2013, 63-68

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

ON EQUALITY OF NEWTON'S FORWARD, NEWTON'S BACKWARD AND LAGRANGE'S INTERPOLATION FORMULA*

ARUN KUMAR ADHIKARY

Indian Statistical Institute, Kolkata, West Bengal, India

ABSTRACT

In this paper Newton's forward interpolation formula is shown to be exactly identical with Newton's backward interpolation formula. When the values of the argument are equidistant, it is also shown that Lagrange's interpolation formula coincides with Newton's forward interpolation formula.

KEYWORDS: Argument, Entry, Forward Interpolation, Backward Interpolation

1. INTRODUCTION

It is quite well-known that when the values of the argument x are equidistant and we are required to interpolate the value of the entry y = f (x) corresponding to a value of x near the beginning of a set of tabulated values, we apply Newton's forward interpolation formula. On the other hand, if we are to interpolate the value of y corresponding to a value of x near the end of a set of tabulated values of x, then we apply Newton's backward interpolation formula. In this paper an attempt has been made to show that Newton's forward interpolation formula and Newton's backward interpolation formula are identical. Also when the values of x are equidistant, Lagrange's interpolation formula coincides with Newton's forward interpolation formula.

2. THE RESULTS

Suppose $x_0, x_1, ..., x_n$ are the given values of the argument x which are equidistant and h is their interval of differencing. Let $y_0, y_1, ..., y_n$ be the corresponding values of the entry y = f (x). Suppose we are required to interpolate the value of y corresponding to a value of x which may lie either at the beginning or at the end or at the middle of the set of the tabulated values.

Then Newton's forward interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots. + \frac{u(u-1)\ldots.(u-n+1)}{n!}\Delta^n y_0$$

where $\Delta y_0 = y_1 - y_0$, $\Delta^2 y_0 = y_2 - 2y_1 + y_0$ etc. are the finite differences of first order, second order etc. and $u = \frac{x - x_o}{h}$.

Again Newton's backward interpolation formula is

*Presented as an Invited Talk in the Section of Mathematical Sciences including Statistics in the 95th Session of Indian Science Congress held at Andhra University, Vishakapatnam during 3-8 January ,2008 .

$$y = y_n + v\Delta y_{n-1} + \frac{v(v+1)}{2!}\Delta^2 y_{n-2} + \ldots + \frac{v(v+1)\ldots(v+n-1)}{n!}\Delta^n y_0$$

where $v = \frac{x - x_n}{h}$ and $\Delta y_{n-1} = y_n - y_{n-1}, \Delta^2 y_{n-2} = y_n - 2y_{n-1} + y_{n-2}$ etc.

Then, we have the following theorem.

Theorem 2.1: Newton's forward interpolation formula = Newton's backward interpolation formula.

Proof: Newton's forward interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0$$

$$= \left[1 + \Delta\binom{u}{1} + \binom{u}{2}\Delta^2 + \ldots + \binom{u}{n}\Delta^n\right] y_0$$

$= (1+\Delta)^u y_0$ since $\Delta^r y_0 = 0$ if $r >$ n.

Newton's backward interpolation formula is

$$y = y_n + v\Delta y_{n-1} + \frac{v(v+1)}{2!}\Delta^2 y_{n-2} + \ldots + \frac{v(v+1)\ldots(v+n-1)}{n!}\Delta^n y_0$$

$$= y_n + v\Delta E^{-1} y_n + \frac{v(v+1)}{2!}\Delta^2 E^{-2} y_n + \ldots + \frac{v(v+1)\ldots(v+n-1)}{n!}\Delta^n E^{-n} y_n$$

$= (1 - \Delta E^{-1})^{-v} y_n$ since $\Delta^n y_0$ is constant and higher order differences are all zero as (n+1) values of y are given

$= (1 - \Delta E^{-1})^{-(u-n)} E^n y_0$ as $v = \frac{x - x_n}{h} = \frac{(x - x_0) - (x_n - x_0)}{h} = u - n$

$= \left[1 - (E-1)E^{-1}\right]^{-(u-n)} E^n y_0$ as $\Delta \equiv E - 1$

$$= \left[1 - (1 - E^{-1})\right]^{-(u-n)} E^n y_0$$

$$= E^{u-n} E^n y_0$$

$$= E^u y_0$$

$= (1+\Delta)^u y_0$, which is Newton's forward interpolation formula. Q.E.D.

Lagrange's interpolation formula is

$$y = \sum_{r=0}^{n} \frac{\psi(x)}{(x - x_r)\psi'(x_r)} y_r$$

where $\psi(x)=\prod_{r=0}^{n}(x-x_r)$ and $\psi'(x_r)$ is the value of $\psi'(x)$ at $x=x_r$.

Then, we have the following theorem.

Theorem 2.2: When the values of the argument x are equidistant, then Lagrange's interpolation formula = Newton's forward interpolation formula.

Proof: As the values of x are equidistant and h is the common interval of differencing, $\psi(x)$ can be written as

$$\psi(x)=\prod_{r=0}^{n}(x-x_r)=(x-x_0)(x-x_1).....(x-x_n)$$

$$= hu.h(u-1)\ldots h(u-n)$$

$$= h^{n+1}u(u-1)....(u-n) \tag{2.1}$$

Also $\psi'(x_r)$ can be written as

$$\psi'(x_r)=(x_r-x_0)(x_r-x_1)....(x_r-x_{r-1})(x_r-x_{r+1})....(x_r-x_n)$$

$$= rh.(r-1)h\ldots h(-h)\ldots(n-r)(-h)$$

$$= r!h^r.(-1)^{n-r}h^{n-r}(n-r)!$$

$$= (-1)^{n-r}h^n r!(n-r)! \tag{2.2}$$

Again $(x-x_r)=(x-x_0)-(x_r-x_0)=hu-hr=(u-r)h$ (2.3)

From (2.1), (2.2) and (2.3), Lagrange's interpolation formula can be written as

$$y=\sum_{r=0}^{n}h^{n+1}\frac{u(u-1)....(u-n)}{(u-r)h}.\frac{1}{(-1)^{n-r}h^n r!(n-r)!}.y_r$$

$$= \sum_{r=0}^{n}\frac{u(u-1)....(u-n)}{(u-r)}\frac{1}{(-1)^{n-r}r!(n-r)!}y_r \tag{2.4}$$

Now Newton's forward interpolation formula can be written as

$$y=\sum_{r=0}^{n}\binom{u}{r}\Delta^r y_0$$

and the coefficient of y_r in Newton's forward interpolation formula is

$$\frac{u(u-1)\ldots(u-r+1)}{r!}-\frac{u(u-1)\ldots(u-r)}{(r+1)!}\binom{r+1}{1}+\frac{u(u-1)\ldots(u-r-1)}{(r+2)!}\binom{r+2}{2}-\ldots\ldots$$

$$+(-1)^{n-r}\frac{u(u-1)\ldots(u-n+1)}{n!}\binom{n}{n-r}$$

$$= \sum_{j=0}^{n-r} \frac{u(u-1)\ldots(u-r-j+1)}{(r+j)!}(-1)^j \binom{r+j}{j}$$

$$= \sum_{j=0}^{n-r} (-1)^j \binom{u}{r+j}\binom{r+j}{j}$$

$$= \sum_{j=0}^{n-r} (-1)^j \frac{u!}{(r+j)!(u-r-j)!}\frac{(r+j)!}{j!r!}$$

$$= \sum_{j=0}^{n-r} (-1)^j \binom{u}{r}\binom{u-r}{j}$$

$$= \binom{u}{r}\sum_{j=0}^{n-r} (-1)^j \binom{u-r}{j}$$

$$= \binom{u}{r}\sum_{j=0}^{n-r} \text{ The coefficient of } t^j \text{ in } (1-t)^{u-r}$$

$$= \binom{u}{r}\sum_{j=0}^{n-r} \text{ Term independent of t in } \frac{(1-t)^{u-r}}{t^j}$$

$$= \binom{u}{r} \text{ Term independent of t in } (1-t)^{u-r}\sum_{j=0}^{n-r}\frac{1}{t^j}$$

$$= \binom{u}{r} \text{ Term independent of t in } (1-t)^{u-r} \frac{\left[\left(\frac{1}{t}\right)^{n-r+1} - 1\right]}{\left(\frac{1}{t}-1\right)}$$

$$= \binom{u}{r} \text{ Term independent of t in } (1-t)^{u-r-1}\left[\frac{1}{t^{n-r}} - t\right]$$

$$= \binom{u}{r} \text{ Term independent of t in } \frac{(1-t)^{u-r-1}}{t^{n-r}}$$

$$= \binom{u}{r} \text{ The coefficient of } t^{n-r} \text{ in } (1-t)^{u-r-1}$$

$$= \binom{u}{r}(-1)^{n-r}\binom{u-r-1}{n-r}$$

$$= (-1)^{n-r}\frac{u!}{r!(u-r)!}\frac{(u-r-1)!}{(n-r)!(u-n-1)!}$$

$$= (-1)^{n-r} \frac{u(u-1)....(u-n)}{(u-r)r!(n-r)!} \quad (2.5)$$

Thus Newton's forward interpolation formula can be written as

$$y = \sum_{r=0}^{n} (-1)^{n-r} \frac{u(u-1).....(u-n)}{(u-r)} \frac{1}{r!(n-r)!} y_r$$

$$= \sum_{r=0}^{n} \frac{u(u-1)....(u-n)}{(u-r)} \frac{1}{(-1)^{n-r} r!(n-r)!} y_r$$

which exactly matches with Lagrange's interpolation formula given by (2.4). Hence the proof.

From Theorem 2.1 and Theorem 2.2, we have the following corollary.

Corollary 2.1: When the values of the argument x are all equidistant, Newton's forward interpolation formula, Newton's backward interpolation formula and Lagrange's interpolation formula are all identical.

Proof: Follows from Theorem 2.1 and Theorem 2.2.

3. A NUMERICAL EXAMPLE

Suppose the values of a function y = f (x) are given for some equidistant values of x as follows:

x	Y
2	16
3	81
4	256
5	625
6	1296

Suppose we want to interpolate the value of y corresponding to x = 2.5, 4.1 and 5.5. Consider the following difference table.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
16				
	65			
81		110		
	175		84	
256		194		24
	369		108	
625		302		
	671			
1296				

It is easy to check that the interpolated values of y corresponding to a value of x obtained by Newton's forward interpolation formula, Newton's backward interpolation formula and Lagrange's interpolation formula are all equal as is evident from the following table.

Table 1

x	Value of y Obtained by		
	Newton's Forward Interpolation Formula	Newton's Backward Interpolation Formula	Lagrange's Interpolation Formula
2.5	39.0625	39.0625	39.0625
4.1	282.5761	282.5761	282.5761
5.5	915.0625	915.0625	915.0625

Thus when the values of the argument are equidistant, there is nothing to discriminate among Newton's forward interpolation formula, Newton's backward interpolation formula and Lagrange's interpolation formula and we may apply any one of the three formulae irrespective of whether x is at the beginning of the table or at the end of the table or at the middle of the tabulated values.

REFERENCE

1. Scarborough, J. B. *Numerical Mathematical Analysis*. Oxford University Press, Oxford Book Co., 1958.